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The integral geometry of random fields has been investigated since the 1980s, and the analytic formulae of the Minkowski functionals (also called Lipschitz-Killing curvatures, shortly denoted LKCs) of their excursion sets on a compact subset S in the n-dimensional Euclidean space have been reported in the specialized literature for Gaussian and student-t random fields. Very recently, explicit analytical formulae of the Minkowski functionals of their excursion sets in the bi-dimensional case ($n = 2$) have been defined on more sophisticated random fields, namely: the *Linearly Combined Gaussian and Student-t*, and the *Skew Student-t* random fields. This paper presents the theoretical background, and gives the explicit analytic formulae of the three Minkowski functionals. Simulation results are also presented both for illustration and validation, together with a real application example on an worn engineered surface.

1 Introduction

The geometrical evolution of the surface topography of random fields avails understanding many important physical phenomena, as example for engineered surfaces. During a wear (generally engineering) process, the surface topography will deform, during time, more or less at different spatial scales. This paper aims at presenting the analytic expressions of the Minkowki fonctionals for the *Linearly Combined Gaussian and Student-t*, and the *Skew Student-t* random fields, and at characterizing the surface roughness,by tackling the integral geometry and the differentiable topology of the random field model. For this aim, the real surface is assumed to be rough, but differentiable almost everywhere. The characterizing functions provide a full description of the spatial content of the excursion sets; in practice they can be used for estimating the model's parameters from real data by fitting the empirical functions with the analytical ones. Secondly, the expected EC is a good estimate of the extreme values (global maxima or minima) of the random field.

The paper is organized as follows. First, a brief review of some basics in the theory of random fields and their integral geometry is presented. In section 3 and 4, the linearly combined Gaussian and Student-t random field, and the Skew student's t random field, and their integral geometry are presented, respectively, including simulation results allowing analytic formulae of the Minkowki fonctionals and their numerical approximations to be compared. Next, in section 5 a real application example on a real worn engineered surfaces is illustrated. The paper ends with a concluding discussion (section 6).

2 Random fields and their integral geometry

This section reviews briefly some basics in the theory of random fields and their integral geometry (see also [10, 1]).

2.1 Random fields and their excursion sets

We will restrict the definition of the random fields on rectangles of $\mathbb{R}^N$ conveniently without loosing generality.

Let $\{Y(x_k)\}_{k=1,...d}$, $x \in S$, $(S \subset \mathbb{R}^N)$ be an arbitrary collection of d random variables on a rectangle S of the $N-$dimensional Euclidean space $\mathbb{R}^N$, for any choice of $x_1, ..., x_d$, with a finite-dimensional distributions $F_{x_1,...,x_d}$ on $\mathbb{R}^d$ defined by:

$$F_{x_1,...,x_d}(h_1, ..., h_d) = \mathbb{P}\{Y_{x_1} \leq h_1, ..., Y_{x_d} \leq h_d\} \tag{1}$$

Then, an $N-$dimensional random field $\{Y(x) : x \in S\}$ is defined by its finite dimensional distribution function at any $x \in S$. The excursion set of a real-valued random field Y in $S \subset \mathbb{R}^N$ over a threshold h, $(h \in \mathbb{R})$, is defined as:

$$E_h(Y, S) = \{x \in S : Y(x) \geq h\} \tag{2}$$

The set E_h includes the random components that result from thresholding Y at the level h.

2.2 Integral geometry of a random field defined on $\mathbb{R}^N$

The geometric properties of the excursion sets are indeed its Euler-Poincaré characteristic and its associated LKCs,which are adressed within the framework of integral geometry.

Let $S = \prod_{i=1}^{N}[0, R_i]$ be a rectangle set in $\mathbb{R}^N$, and suppose J be the $k-$dimensional face of S, $(J \in \partial_k S)$, such that:

$$J = \{x \in \mathbb{R}^N : x_i = \epsilon_i R_i,\ \text{if}\ i \notin \sigma(J),\ \text{or}\ 0 \leq x_i \leq R_i, \text{if}\ i \in \sigma(J)\} \tag{3}$$

where $\sigma(J) \subseteq \{1, ..., N\}$, and $\epsilon(J) = \{\epsilon_i, i \notin \sigma(J)\}$ is a sequence of $N-k$ zeros and ones, where $\epsilon_i \in \{0, 1\}$. Then, based on Steiner's formulae ([7, 9]), the $j-$th dimensional LKCs, $(j > 0)$, of S are defined as:

$$\mathcal{L}_j(S) = \sum_{J \in \mathcal{O}_k} vol_j(J) \tag{4}$$

where vol_j stands for the $j-$th dimensional volume of J, and $\mathcal{O}_k$ denotes the $\binom{N}{k}$ elements of $\partial_k S$ including the spatial origin.

In the two-dimensional case $\mathbb{R}^2$, $\mathcal{L}_1(S)$ is the half-boundary length of S, and $\mathcal{L}_2(S)$ measures the area of S. $\mathcal{L}_0(S)$ is the Euler-Poincaré characteristic of S, namely $\chi(S)$.

Now, suppose a real-valued stationary random field, denoted by Y, is defined on S in terms of n independent, identically distributed, stationary Gaussian random fields, G_d, $(d = 1, ..., n)$, with zero means, and unit variance. Let $\Lambda = Var\left(\partial G_d(x)/\partial x\right)$ be a $(N \times N)$ matrix of the second order spectral moments of G_d. This yields scaling the space S by $\Lambda^{1/2}$, then, the $j-$th dimensional LKCs of S will be rewritten as:

$$\mathcal{L}_j(S) = \sum_{J \in \mathcal{O}_k} [\det(\Lambda_J)]^{1/2} vol_j(J) \tag{5}$$

where Λ_J is the $k \times k$ second-order spectral moments associated with the $k-$dimensional face J.

2.3 $k-$th dimensional Euler characteristic densities

When Y is regular on S, (i.e., twice differentiable on S and on ∂S the boundaries of S), the $j-$th dimensional LKCs of its excursion set E_h at level h are defined, based on Hadwiger's theorem [6], as[1]:

$$\mathbb{E}\left\{\mathcal{L}_j(E_h(Y, S))\right\} = \sum_{k=0}^{N-j} \left[\begin{array}{c} j+k \\ k \end{array}\right] \mathcal{L}_{k+j}(S)\rho_k(h) \tag{6}$$

where $\rho_k(h)$, $(k = 1, ..., N)$, are the $k-$th dimensional Euler characteristic (EC) densities of E_h at h. They do not depend on the geometry of S, but on the model of Y. Generally, they are calculated using Morse theory [8] which states, for $k > 0$:

$$\rho_k(h) = \mathbb{E}\left(1_{\{Y \geq h\}} \det(-\ddot{Y}_k) | \dot{Y}_k = 0\right) \mathbb{P}(\dot{Y}_k = 0) \tag{7}$$

[1] $\left[\begin{array}{c} n \\ k \end{array}\right] = \left(\begin{array}{c} n \\ k \end{array}\right) \frac{\omega_n}{\omega_k \omega_{n-k}}$ where $\omega_k = \frac{\pi^{k/2}}{\Gamma(k/2+1)}$ is the volume of the unit k-dimensional ball in $\mathbb{R}^n$.

where $\dot{Y}_k$ denotes the first-order partial derivatives of the first $k-$th components on S, and $\ddot{Y}_k$ is the matrix of the second-order partial derivatives of the first $k-th$ components on S.

3 Linearly combined Gaussian and Student-t random field, and its integral geometry

In the following, the expected EC densities are derived for a isotropic GT^{ν}_{β} random field obtained from the linear combination of two independent Gaussian random field and a student-t random field with ν degrees of freedom and $\beta > 0$ ([2]).

3.1 GT^{ν}_{β} random field

On a subset S of $\mathbb{R}^N$, if any arbitrary D-dimensional random vector, $(Y(x_1), ..., Y(x_D))^t$ has a GT^{ν}_{β} multivariate distribution, then for any $x \in S$, $Y(x)$ will define GT^{ν}_{β} random field, which yields to the following definition :

Definition 1 (GT^{ν}_{β} random field)**.** *Let G be a stationary, not necessarily isotropic, Gaussian random field on a compact subset $S \subset \mathbb{R}^N$ with zero mean, $\mu_G = 0$, and unit variance $\sigma^2_G = 1$. Let T^{ν} be a homogeneous student-t random field with ν degrees of freedom, independent of G. Then, the sum given by:*

$$Y(x) = G(x) + \beta T^{\nu}(x), \quad \beta \in \mathbb{R}^* \tag{8}$$

defines a stationary GT^{ν}_{β} real-valued random field with ν degrees of freedom.

3.2 Expected EC densities for GT^{ν}_{β} random fields

Theorem 1. *The j-th dimensional EC densities, $\rho_j(.)$, $j = 0, 1, 2$ for a isotropic GT^{ν}_{β} random field on $\mathbb{R}^2$, with ν degrees of freedom, $\nu \geq 2$, and $\beta > 0$, are* *defined for a given level h by:*

(i) $\rho_0(h) = \mathbb{P}[Y \geq h] = \mathbb{E}[\mathbb{P}[\beta T^\nu \geq h - G|G]]$

$$= \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{(2\pi)\beta\sqrt{\nu/2}\Gamma\left(\frac{\nu}{2}\right)} \int_h^\infty \int_{-\infty}^\infty \left(1 + \frac{(h-g)^2}{\beta^2\nu}\right)^{-\frac{\nu+1}{2}} e^{-g^2/2} dg$$

(ii) $$\rho_1(h) = \frac{\lambda^{1/2}}{(2\pi)^{3/2}} \int_{-\infty}^\infty \left(1 + \frac{(h-g)^2}{\beta^2\nu}\right)^{-\frac{\nu-1}{2}} e^{-g^2/2} dg$$

$$+ \frac{\lambda_G^{1/2}\Gamma\left(\frac{\nu+1}{2}\right)}{(2\pi)^{3/2}\Gamma\left(\frac{\nu}{2}\right)\beta\sqrt{\nu}} \int_{-\infty}^\infty \left(1 + \frac{(h-g)^2}{\beta^2\nu}\right)^{-\frac{\nu+1}{2}} e^{-g^2/2} dg$$

(iii) $$\rho_2(h) = \frac{2^{\frac{1}{2}}\lambda\Gamma\left(\frac{\nu+1}{2}\right)}{(2\pi)^2\Gamma\left(\frac{\nu}{2}\right)} \int_{-\infty}^\infty \frac{(h-g)}{\beta\sqrt{\nu}} \left(1 + \frac{(h-g)^2}{\beta^2\nu}\right)^{-\frac{\nu-1}{2}} e^{-g^2/2} dg$$

$$+ \frac{\lambda_G\Gamma\left(\frac{\nu+1}{2}\right)}{(2\pi)^2\Gamma\left(\frac{\nu}{2}\right)\beta\sqrt{\nu/2}} \int_{-\infty}^\infty g \left(1 + \frac{(h-g)^2}{\beta^2\nu}\right)^{-\frac{\nu+1}{2}} e^{-g^2/2} dg$$

where $\Lambda_G = \lambda_G I_2$ is the second spectral moments matrix of G, and $\Lambda = \lambda I_2$ is the second spectral moments matrix associated with T^ν.

The Figure 1 presents a simulation example of GT_β^ν random field, with $\nu = 5$ degrees of freedom and $\beta = 0.5$. The expected LKCs of its excursion sets are compared with the simulated ones for illustration in (Fig. 1(b)).

4 Skew student's t random field and its integral geometry

In this section, the skew student's t random field is defined on S by its finite-dimensional distributions given in [5].

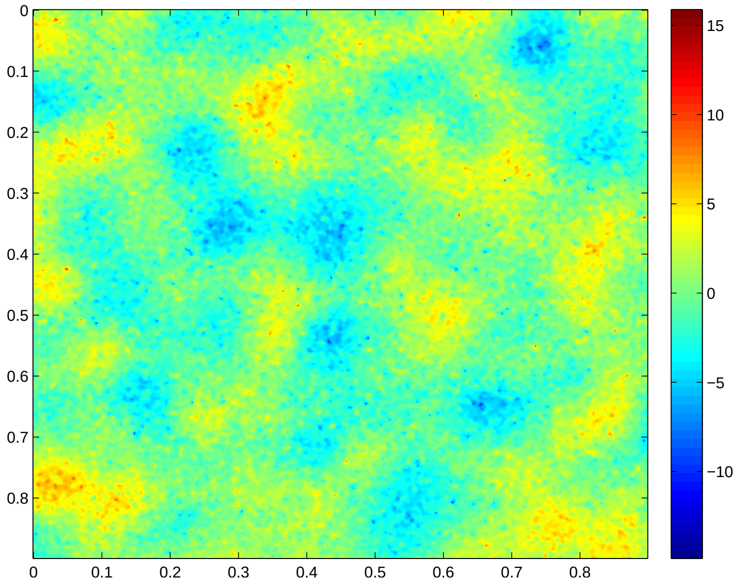
4.1 Skew student's t random fields

Let $Z, G_1, ..., G_\nu$ be independent, identically distributed, stationary Gaussian random fields with zero mean, unit variance, and $\Lambda = \mathrm{Var}(\partial Z/\partial x) = \mathrm{Var}(\partial G_k/\partial x)$, $k = 1, ..., \nu$. Let $z \sim Normal(0, 1)$ be a Gaussian random variable independent of $Z(x)$, $G_1(x), ..., G_\nu(x)$ and δ be a real value such that $\delta^2 < 1$. Then, a real-valued skew student's t random field, $Y(x)$, with ν degrees of freedom and skewness index δ, is defined, at any $x \in S$, as:

$$Y(x) = \frac{\delta|z| + \sqrt{1-\delta^2}Z(x)}{\left(\sum_{k=1}^\nu G_k^2(x)/\nu\right)^{1/2}} \tag{9}$$

The marginal distribution of Y, denoted p_Y, at any fixed x of S is the known skew student's t probability density function [5]:

$$p_Y(h) = 2t_\nu(h)T\left(\alpha h\sqrt{\frac{\nu+1}{\nu+h^2}}; \nu+1\right) \tag{10}$$



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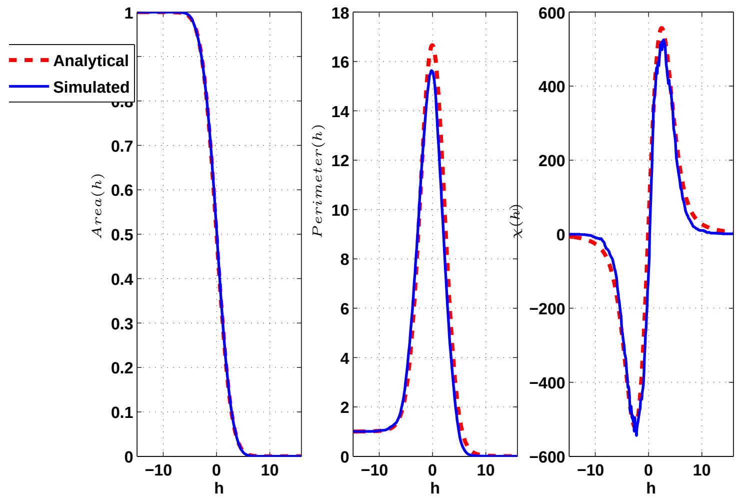


(b)

Figure 1: (a) Linearly combined Gaussian and Student-t random field with 5 degrees of freedom and $\beta = 0.5$. (b) The numerical and the analytical Lipschitz-Killing Curvatures, A, C and χ, respectively.

where $\alpha = \delta/\sqrt{1-\delta^2}$, t_ν is the standard student's t probability density function of ν degrees of freedom, and $T(.;\nu+1)$ is the student's t cumulative distribution function of $\nu+1$ degrees of freedom at $\alpha h\sqrt{(\nu+1)/(\nu+h^2)}$

4.2 The expected EC densities for the skew student's t excursion sets

In the following, we derive the expected EC densities $\rho_k(h)$ for the skew-t random field $Y(x) \in \mathbb{R}^2$ which are proved in previous works ([3, 4]). Figure 2 illustrates a simulation example.

Theorem 2. *The analytical formulae of the EC densities, $\rho_j(.)$, $j = 0,1,2$, for a stationary skew$-t$ random field of ν degrees of freedom , $(\nu > 2)$, and skewness index δ on $\mathbb{R}^2$, at a given level h are:*

$$(i) \quad \rho_0(h) = 2\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\nu\pi}\Gamma\left(\frac{\nu}{2}\right)}\int_h^\infty \left(1+\frac{y^2}{\nu}\right)^{-\frac{\nu+1}{2}} T_1\left(\alpha y\sqrt{\frac{\nu+1}{y^2+\nu}};\nu+1\right)dy$$

$$(ii) \quad \rho_1(h) = \frac{2}{2\pi}(1-\delta^2)\left(1+\frac{h^2}{\nu(1-\delta^2)}\right)\left(1+\frac{h^2}{\nu}\right)^{-\frac{\nu+1}{2}} T_1\left(\alpha h\sqrt{\frac{\nu+1}{\nu+h^2}};\nu+1\right)$$

$$(iii) \quad \rho_2(h) = \frac{2}{(2\pi)^{3/2}}\frac{(1-\delta^2)^{\frac{1}{2}}\Gamma\left(\frac{\nu+1}{2}\right)}{\left(\frac{\nu}{2}\right)^{\frac{1}{2}}\Gamma\left(\frac{\nu}{2}\right)}h\left(1+\frac{h^2}{\nu(1-\delta^2)}\right)\left(1+\frac{h^2}{\nu}\right)^{-\frac{\nu+1}{2}}$$
$$T_1\left(\alpha h\sqrt{\frac{\nu+1}{\nu+h^2}};\nu+1\right)\left[1-2\delta(1-\delta^2)^{\frac{1}{2}}\frac{\pi^{-\frac{1}{2}}\nu^{\frac{1}{2}}\Gamma\left(\frac{\nu}{2}\right)}{\Gamma\left(\frac{\nu+1}{2}\right)h}\left(1+\frac{h^2}{\nu(1-\delta^2)}\right)^{-\frac{1}{2}}\right]$$

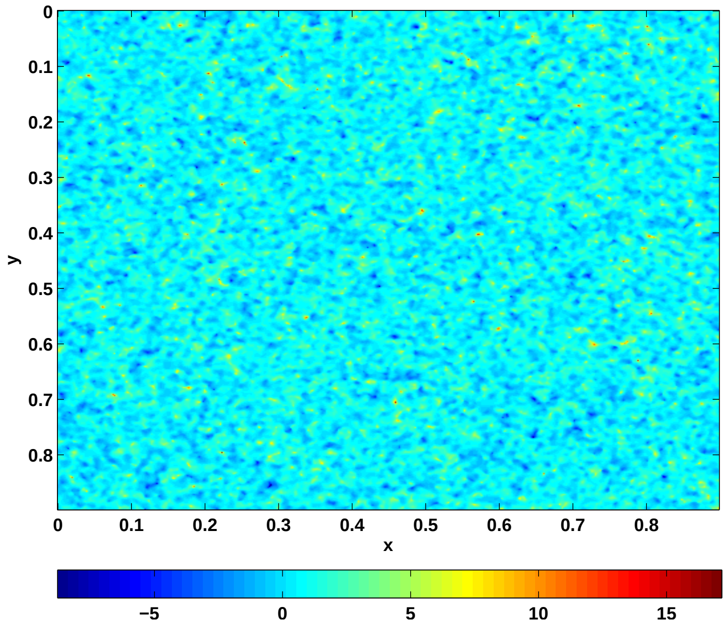
where $\alpha = \delta/\sqrt{1-\delta^2}$.

5 Real application examples

The two stochastic models has been tested on a real 3-D microstructured rough surface of a UHMWPE (Ultra High Molecular Weight Polyethylene) component.The surface has been measured by a non-contact white light interferometric optical system, (Bruker nanoscope Wyko® NT 9100). The height measurements are discretized over a two dimensional sample S of $0.86\times0.86mm^2$ size and with spacing resolution $1.8\mu m$ in both x and y directions, outcoming on a lattice of 480×640 points, see Figure 4.

5.1 GT^ν_β random field

The rough asperities are assumed to have GT^ν_β distribution, thus a GT^ν_β random field is defined on S. The Gaussian random field is assumed to be anisotropic where as the student-t is homogeneous over S. The anisotropy is estimated from the correlation function which gives $\lambda_{G_x} = 15\text{mm}^{-2}$, $\lambda_{G_y} = 117\text{mm}^{-2}$.



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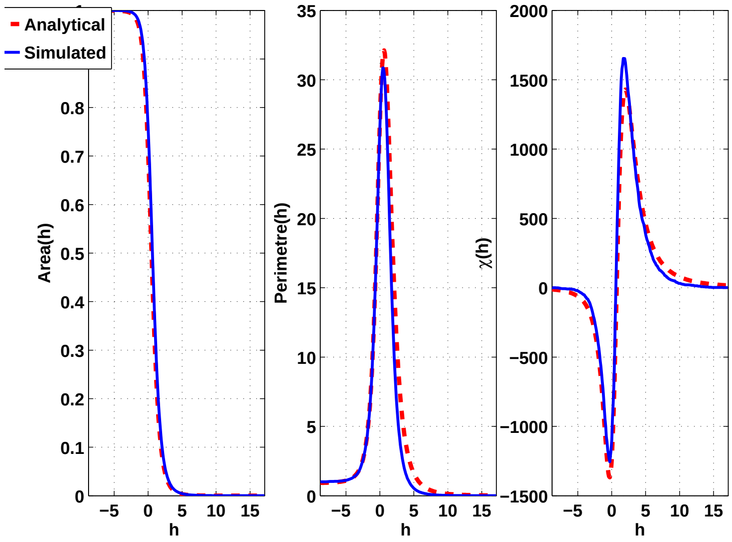

(b)

Figure 2: (a) Isotropic skew student's t random field with 5 degrees of freedom and skewness index $\delta = 0.5$. (b) The numerical and the analytical Lipschitz-Killing Curvatures, A, C and χ, respectively.

The student-t random field parameters are estimated from minimizing the error between the empirical and the expected Euler-Poincaré characteristic function, which yields to $\nu = 5$, $\beta = 0.2$ and $\lambda_X = 190.3\text{mm}^{-2}$. Figure 3 shows the fitting result between the expected and the empirical characteristic functions of the excursion sets of the GT^{ν}_{β} random field and the real surface. Although the results show that the approximation is close to the real measurements. The model can not describe the effect of some statistical parameters such as the third order moment which defines the skewness of the heights distribution which becomes a significant characteristic of worn surfaces.

5.2 Skew student's t random field and its integral geometry

Figure 4(b) shows a result of fitting the LKCs computed numerically for a real surface and the analytical ones for a skew-t random field with 8 degrees of freedom and skewness index $\delta = -0.7$. The results show the ability to use this stochastic model to describe the roughness evolution during the wear process.

6 Concluding discussion and future work

This paper has highlighted the importance of integral geometry, and in particuler of expected Euler Poincaré characteristic for two types of random fields, namely: the *Linearly Combined Gaussian and Student-t*, and the *Skew Student-t* random fields. The aim was to model the topography of rough surfaces during wear process and skew-t demonstrated better results. The authors aim at applying this model to quantify the roughness evolution during wear time.

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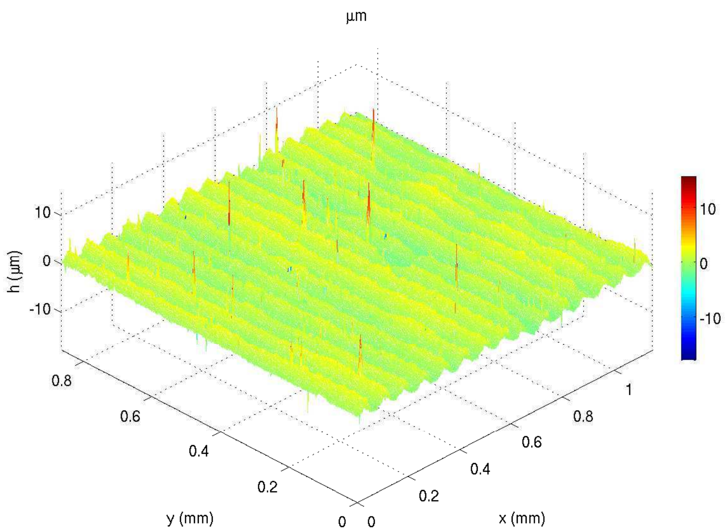

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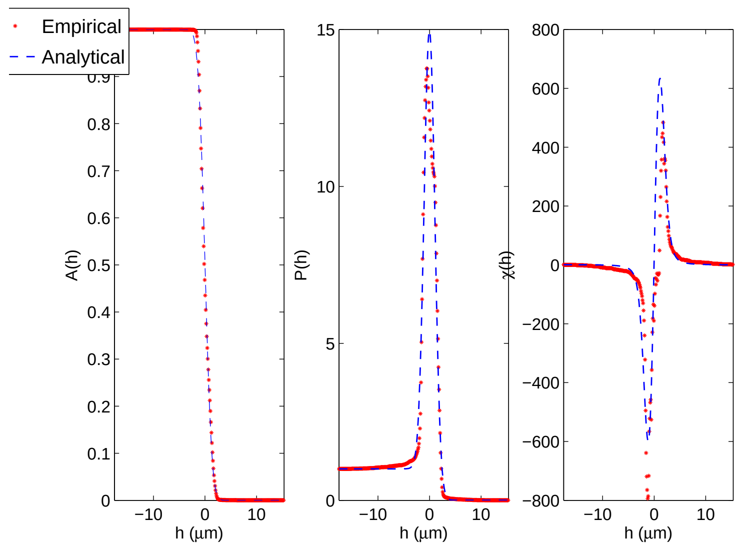
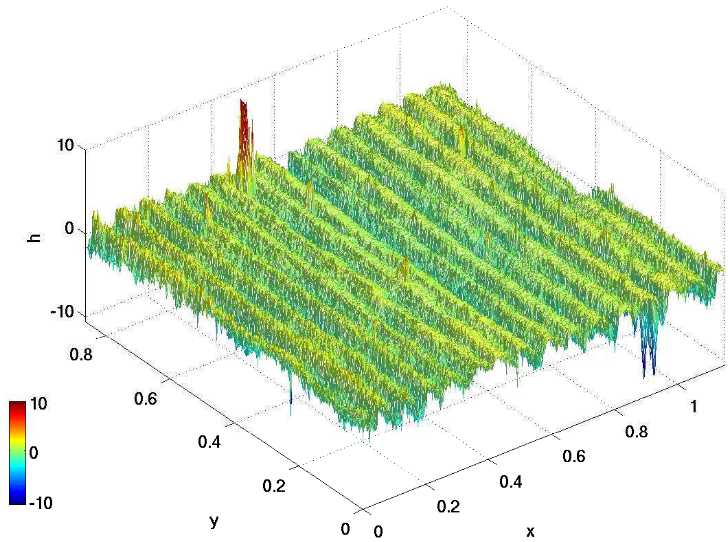


(b)

Figure 3: Real application example. (a) A real 3D observation of a rough surface of a plastic material (UHMWPE). (b) The expected and the empirical Minkowski functions, $A(h), Per(h)$ and $\chi(h)$, respectively, of the excursion sets of the random field $GT^5_{0.2}$ and the real 3D surface.



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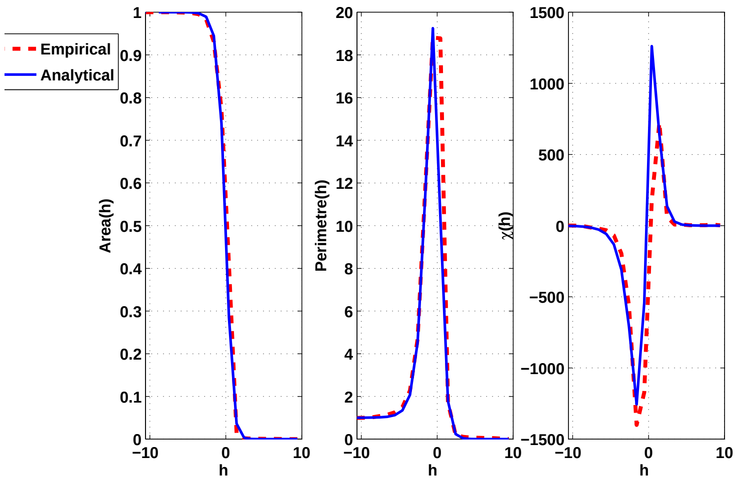

(b)

Figure 4: An application example, acquired by a non-contact white light interferometric optical system. (a) A real 3D surface roughness topography digitized on a lattice of 480 points with a spatial sampling steps equal to $1.8\mu m$ in x and y directions. (b) Fitting the empirical and the analytical LKCs for the real surface and the skew student's t with 8 degrees of freedom and skewness index $\delta = -0.7$ and $|\Lambda|^{1/2} = 4 \cdot 10^4 \mathrm{mm}^{-2}$, respectively.

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